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Can The Order of Convergence Be Higher Than the Number of Function Values Used? Part (1)

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Abstract

A nonlinear algebraic equation solver with the 5^{th} order of convergence but uses only 4 function values in eash iteration, is described, and demonstrated with an numerical example.

PROOF FOR THE ORDER OF CONVERGENCE

Existing methods for solving a regular non-linear equation $f(\xi) = 0$ have the property that the number of function calls required in an iteration cycle equals the order of convergence. For example, a simple iteration $\xi_{n+1} = f(\xi_n) + \xi_n$ uses one function value and is first order convergent; a bi-section method $\xi_{n+1} = 0.5(\xi_n + \xi_{n-1})$ which evaluates $f(\xi_n)$ at each iteration to determine which line-section the solution falls into, is also first order; a Newton-Raphson method $\xi_{n+1} = \xi_n - f(\xi_n)/f'(\xi_n)$ which calculates two function values $f(\xi)$ and $f'(\xi)$ is second order. The ratio $\mathbf{R}$ between the order of convergence and the number of function values required for an iteration is always 1.

The following new solution method has $\mathbf{R = 5/4}$, so it is faster than other existing iterative methods. There are two steps with this method. The first step is the conventional Halley's method[1]. Let ξ_n be the n^{th} guess for the root. One solves the equation

$$f(\xi_n) + f'(\xi_n)\delta + \frac{1}{2} f''(\xi_n)\delta^2 = 0, \tag{1}$$

For consistency with the Newton-Raphson method when the quadratic term vanishes, the root is chosen to be

$$\delta = -\frac{f'(\xi_n)}{f''(\xi_n)} \left(1 - \sqrt{1 - \frac{2f(\xi_n)f''(\xi_n)}{(f'(\xi_n))^2}} \right).$$

Note that the solution of eq. (1) implies $f(\xi_n + \delta) = O(\delta^3)$. This step uses three function calls.

The next step uses one more function call to gain two more orders of convergence in addition to that of Halley's original irrational formula. One adds a term $f(\xi_n + \delta)$ into eq.(1), and solves

$$f(\xi_n) + f(\xi_n + \delta) + f'(\xi_n)\Delta + \frac{1}{2}f''(\xi_n)\Delta^2 = 0. \tag{2}$$

The solution is similar to what is obtained in the first step

$$\Delta = -\frac{f'(\xi_n)}{f''(\xi_n)}\left(1 - \sqrt{1 - \frac{2(f(\xi_n) + f(\xi_n + \delta))f''(\xi_n)}{(f'(\xi_n))^2}}\right).$$

Finally, let $\xi_{n+1} = \xi_n + \Delta$ for completion of the current iteration cycle.

One computes only four function values $f(\xi_n), f'(\xi_n), f''(\xi_n)$, and $f(\xi_n + \delta)$. However, the above scheme is fifth order convergent as shown below.

From a Taylor's Expansion one obtains

$$f(\xi_n + \Delta) = f(\xi_n) + f'(\xi_n)\Delta + \frac{1}{2}f''(\xi_n)\Delta^2 + \frac{1}{6}f^{[3]}(\xi_n)\Delta^3 + O(\Delta^4).$$

From eq. (2), it is equal to $-f(\xi_n + \delta) + f^{[3]}(\xi_n)\Delta^3/6 + O(\Delta^4)$. However, from eq. (1) and the Taylor expansion of $f(\xi_n + \delta)$, the above estimate becomes

$$\frac{1}{6}f^{[3]}(\xi_n)(\Delta^3 - \delta^3) + O(\Delta^4 - \delta^4) = (\Delta - \delta)O(\Delta^2, \delta^2).$$

By subtracting eq. (1) from eq. (2) one arrives at

$$(\Delta - \delta)(f'(\xi_n) + O(\delta)) = -f(\xi_n + \delta) = O(\delta^3). \tag{3}$$

It tells us that Δ and δ are of the same order and

$$(\Delta - \delta) = O(\delta^3).$$

One clearly sees that

$$f(\xi_{n+1}) = f(\xi_n + \Delta) = O(\Delta^5).$$

Therefore the method is fifth order convergent, however employ only *four* function values.

A NUMERICAL EXAMPLE

We take an simple equation $cos(x) = x$ to demostrate the efficiency of the proposed scheme. Its numberical solution is $x^* = 0.73908513321516.....$

Let $f = x - cos(x)$ one has $f' = 1 + sin(x)$ and $f'' = cos(x)$. We take an innocent initial guess that $x_0 = 0$.

Now perform the first iteration, by first solving $f(x_0) + f'(x_0)\delta + \frac{1}{2}f''(x_0)\delta^2 = 0$, or

$$-1 + \delta + \delta^2/2 = 0$$

one finds that (take the first 25 digits)

$$\delta = 0.7320508075688772935274463 4...$$

Then perform the second step of this iterstion by solving

$$f(x_0) + f(x_0 + \delta) + f'(x_0)\Delta + \frac{1}{2}f''(x_0)\Delta^2 = 0$$

to 25 digits to obtain

$$\Delta = 0.738823974649922658398622 70...$$

Therefore $x_1 = x_0 + \Delta = \Delta$ here. Be aware that this is off the exact solution by only 2×10^{-4}.

Now perform the second iteration by first solving $f(x_1)+f'(x_1)\delta+\frac{1}{2}f''(x_1)\delta^2 = 0$ to obtain that

$$\delta = 0.00026115856404338119688100463655...$$

Now practice the second half of this iteration by solving

$$f(x_1) + f(x_1 + \delta) + f'(x_1)\Delta + \frac{1}{2}f''(x_1)\Delta^2 = 0,$$

one obtains

$$\Delta = 0.00026115856523798324029 58...$$

which makes

$$x_2 = x_1 + \Delta = 0.739085133215160641638918505,$$

and the value of the equation is now -2.74365×10^{-20}. Because the value of f' is order $O(1)$, the accuracy of the root must also be $O(10^{-20})$.

We clearly observed that, with one iteration, 4 digit accuracy is obtained and with two iterations, one obtains 20 digits of accuracy and we have shown the order of convergency is $20/4 = 5$, the same as mathematically proven. However, only $four$ function values are used in each iteration that $f(x), f'(x), f''(x)$, and $f(x+\delta)$.

REFERENCE

[1]. Weisstein, Eric W. "Halley'sMethod." From *MathWorld* - A Wolfram Web Resource.

[2]. Press W. H., Teukolsky S. A., Vetterlingg W. T., and Flannery B. P., *Numerical Recipes in C,* second edition, Cambridge University Press, (1997).